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The Vlasov–Poisson System with Infinite Mass and Energy

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This paper deals with solutions to the Vlasov–Poisson system with an infinite mass. The solution to the Poisson equation cannot be defined directly because the macroscopic density is constant at infinity. To solve this problem, we decompose the solution to the kinetic equation into a homogeneous function and a perturbation. We are then able to prove an existence result in short time for weak solutions to the equation for the perturbation, even though there are no a priori estimates by lack of positivity.

KEY WORDS: Vlasov–Poisson equation; infinite mass; stability.

1. INTRODUCTION

We consider the motion of an infinite number of particles interacting through an electrostatic or gravitational potential in the whole space. The aim of this paper is to investigate the behaviour of the system when the density ρ of particles is a non-vanishing constant at infinity. This implies that the total mass or charge and the kinetic energy of the system are infinite.

This question arises for instance as an approximation of a large system of particles in which we are only interested in what is happening in the center. In this case, it is natural to assume that the system is infinite in size and thus total mass, since the typical length scale that we want to consider is very small in comparison to the scale of the system. Moreover for

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numerical simulations in particular, it is certainly less costly to do this approximation.

The first difficulty is, of course, to define the dynamics since, a priori, the forces acting on the particles are diverging. This problem is solved by decomposing a solution into a particular function which is a solution only formally and a perturbation. The density for the perturbation can then be assumed to decrease at infinity, at the initial time.

The main result of the paper is to show that the corresponding dynamics is stable for at least a short period of time. This means, more precisely, that the density of the perturbation still nicely decreases for a short time. It ensures that no infinite amount of mass can be added to the perturbation due to the influence of the interactions.

We use here a mean field approximation for the dynamics. As a consequence it is described by the Vlasov–Poisson system on the probability distribution $f(t, x, v)$ in the phase space of the particles.

$$\begin{cases} \dfrac{\partial}{\partial t} f + v \cdot \nabla_x f + \nabla U \cdot \nabla_v f = 0, \qquad t \in \mathbb{R}_+, \qquad (x, v) \in \mathbb{R}^6 \\ \Delta U = \alpha \rho \\ \rho(t, x) = \displaystyle\int_{\mathbb{R}^6} f(t, x, v)\, dv \\ f(t=0, x, v) = f^0(x, v) \end{cases} \tag{1.1}$$

The coefficient α depends on the nature of the interaction: $\alpha = 1$ for the electrostatic potential and $\alpha = -1$ for the gravitational potential. The results presented here can also be proved in dimensions one or two with some slight modifications but, for the sake of simplicity, we restrict ourselves to three dimensions in space.

The Poisson equation (second line of (1.1)) has a uniquely defined solution by convolution when ρ is in some L^p with p between 1 and 3. However for more general ρ with no precise decay at infinity, we cannot use the convolution formula or demand any decay of the solution to get uniqueness. Therefore, following the approach initiated by G. Rein and A. D. Rendall (see ref. 22), the solution to the Poisson equation with a constant $c(t)$ plus a function ρ in L^p $(1 \leqslant p \leqslant 3)$ as right-hand side is defined by

$$U(t, x) = \alpha \frac{c(t)}{2} |x|^2 - \frac{\alpha}{4\pi} \frac{1}{|x|} \star \rho(t, x) \tag{1.2}$$

Up to a constant, the first part in U is the potential created inside a ball centered at the origin by charges uniformly distributed in this ball with a concentration c and the condition that U vanishes on the boundary of the ball.

Consequently, it is natural to decompose the solution to (1.1) into a homogeneous solution plus a perturbation which vanishes at infinity. If we denote by $c(t)$ the associated macroscopic density, the homogeneous solution F satisfies

$$\frac{\partial}{\partial t} F + v \cdot \nabla_x F + \alpha c(t)\, x \cdot \nabla_v F = 0 \tag{1.3}$$

Following ref. 22, we consider a solution of the form $F(a(t)\, x + b(t)\, v)$ with F a non-negative and differentiable function whose integral (over $\mathbb{R}^3$) is normalized to 1. In this case, F satisfies the Vlasov–Poisson system under the conditions

$$\begin{cases} \dot{a}(t) + \dfrac{\alpha}{b^2} = 0, \qquad \dot{b}(t) + a(t) = 0 \\ c(t) = b^{-3} \end{cases} \tag{1.4}$$

Finally, when combining (1.1) with (1.3) we obtain for the perturbation f

$$\begin{cases} \dfrac{\partial}{\partial t} f + v \cdot \nabla_x f + (\alpha c(t)\, x + E) \cdot \nabla_v f = -bE \cdot \nabla F(ax + bv) \\ E(t, x) = +\alpha \dfrac{x}{|x|^3} \star \rho(t, x), \qquad \rho = \displaystyle\int f(t, x, v)\, dv \\ f(t=0, x, v) = f^0(x, v) \end{cases} \tag{1.5}$$

At the end of the paper, we will only consider initial data f^0 such that

$$f^0 + F(a(0)\, x + b(0)\, v) \geqslant 0 \tag{1.6}$$

We first state precisely the existence and behaviour of the solutions to (1.4). In the electrostatic case, we have the lemma (proved in Section 2).

Lemma 1.1. Given any initial data $a(0)$ and $b(0) > 0$, there exists a unique solution (a, b), defined on all $[0, \infty[$, to the system (1.4) with $\alpha = 1$. Moreover

(i) If $a(0) \geqslant 0$ then there exists t_0 such that $a(t_0)=0$ and a is decreasing.

(ii) If $a(0) < 0$ then it remains negative for all times.

As for the gravitational case, the following result was proved in ref. 22.

Lemma 1.2. There exists a unique solution (a, b) to (1.4) with $\alpha = -1$ on the maximal interval of existence $[0, T[$ with

(i) if $a(0) \leqslant -\sqrt{2b^{-1}(0)}$, $T = +\infty$, and a remains negative,

(ii) if $a(0) > -\sqrt{2b^{-1}(0)}$ then $T < \infty$, $\lim_{t\to T} b(t)=0$, and a becomes positive if it was not already so.

For both the electrostatic and the gravitational case, it is possible to prove the existence of weak solutions to Eq. (1.5) locally in time, which means on a time interval depending on the size of the initial data.

Theorem 1.1 (local weak solutions). We assume that $F \geqslant 0$, $F \in \mathrm{L}^\infty(\mathbb{R}^3)$, $f^0 \in \mathrm{L}^\infty(\mathbb{R}^6)$, that the solution (a, b) to (1.4) exists on $[0, T]$, and that for some $k > 5$

$$
\begin{aligned}
&(1+|\xi|^k)\, F(\xi) \in \mathrm{L}^1(\mathbb{R}^3), \qquad E^0 \in \mathrm{L}^2(\mathbb{R}^3)\\
&\int_{\mathbb{R}^6} (1+|ax+bv|^k)\, |f^0|^2\, (x, v)\, dx\, dv < +\infty
\end{aligned}
\tag{1.7}
$$

then there exists $0 < t^* \leqslant T$ and $f \in \mathrm{L}^\infty([0, t^*] \times \mathbb{R}^6)$ a solution in distributional sense to (1.5) such that

$$
\begin{aligned}
&\int_{\mathbb{R}^6} (1+|ax+bv|^k)\, |f|^2\, (t, x, v)\, dx\, dv \in \mathrm{L}^\infty([0, t^*])\\
&\rho(t, x) \in \mathrm{L}^\infty([0, t^*], \mathrm{L}^{(k'+3)/3}(\mathbb{R}^3)) \qquad \text{for} \quad 3 < k' \leqslant k\\
&E(t, x) \in \mathrm{L}^\infty([0, t^*], \mathrm{L}^2(\mathbb{R}^3))
\end{aligned}
\tag{1.8}
$$

Remarks

1. The estimate on ρ gives an L^p estimate for the field E, namely E belongs to $\mathrm{L}^\infty([0, t^*], \mathrm{L}^p(\mathbb{R}^3))$ with $2 \leqslant p \leqslant 3\,\frac{k+3}{6-k}$ or $+\infty$ if $k > 6$.

2. The kernel $|ax+bv|$ here plays the role of a dispersion estimate like the kernel $|x-vt|$ in the usual Vlasov–Poisson system. It is already known

that existence results for Vlasov systems can be obtained with moments based on such kernels (see refs. 6 or 19), or for Boltzmann equation (see ref. 18).

Although we do not know whether weak solutions exist globally in time, we can show a conditional existence result for strong solutions to the system (1.5). This is made precise by the following theorem, which studies the propagation of moments. Let us first define (γ, δ), the solution to

$$\begin{aligned} \dot{\gamma}(t) &= -\delta(t), \qquad & \dot{\delta}(t) &= -\alpha c(t)\, \gamma(t) \\ \gamma(0) &= 0, & \delta(0) &= 1 \end{aligned} \tag{1.9}$$

Theorem 1.2 (strong solutions). With the hypothesis of Theorem 1.1, assume also that $\delta(t)$ remains positive and bounded on $[0, T]$, and that for $k_0 > 3$ and for all $0 \leqslant k < k_0$

$$\begin{aligned} & F \in L^1 \cap L^3(\mathbb{R}^3), \qquad |\xi|^k\, |\nabla F| \in L^1 \\ & f \in L^\infty([0, T], L^1(\mathbb{R}^6)), \qquad E \in L^1([0, T] \times \mathbb{R}^3) \\ & \int_{\mathbb{R}^6} |a(0)\, x + b(0)\, v|^k\, |f^0(x, v)|\, dx\, dv < +\infty \end{aligned} \tag{1.10}$$

then all these moments are propagated, for any $0 \leqslant k < k_0$

$$\int_{\mathbb{R}^6} |a(t)\, x + b(t)\, v|^k\, |f|\,(t, x, v)\, dx\, dv \in L^\infty([0, T]) \tag{1.11}$$

Remarks

1. This theorem is conditional on the existence of some nice weak solutions. We require that the solution be in L^1, and so be the force field E which it creates. Unfortunately, we cannot prove either of these conditions. The natural scaling for the field E is $L^{3/2}$, but here, since we may assume that the integral of f is initially zero (and this property is trivially preserved in time), the L^1 norm of E is bounded by the first moment in x of f, $\int |x| \cdot |f|\, dx\, dv$. For the original Vlasov–Poisson system (1.3), it is known that this moment in x remains bounded if it was initially. The equivalent of our hypotheses for this theorem, for the original Vlasov–Poisson system, are then an easy consequence of the theory of existence of weak solutions; such a theory is missing for the system (1.5).

2. The assumption on δ is automatically satisfied if $\alpha=1$. This assumption is needed to control the characteristics in the proof.

3. For the original Vlasov–Poisson system, it is known that all moments of order k larger than 2 are propagated (see ref. 7), instead of 3 for this system.

4. We obtain strong solutions (a representation of the solution with the characteristics) to the system (1.5) when $k_0 \geqslant 6$ because in this case E belongs to L^∞ in x.

The Lemma 1.1 is proved in Section 2. Section 3 is devoted to the proof of Theorem 1.1 and Section 4 to Theorem 1.2.

The systems (1.3) or (1.4) and (1.5) were already derived in ref. 22. G. Rein and A. D. Rendall then performed a change of variables to get rid of the explicit x in the system (1.5), and afterwards they studied the existence of classical solutions in a periodic domain in the x variable. Here we are interested in the problems arising from the infinite mass. Therefore we do not work in a periodic domain and we face different issues.

On the other hand, for finite mass, the existence of weak solutions to the Vlasov–Poisson system (1.1) is well known since A. A. Arsenev,[1] and stronger solutions were studied by J. Batt and G. Rein,[4] E. Horst,[14, 15], K. Pfaffelmoser,[20] J. Schaefer,[23] P. L. Lions and B. Perthame[17] and I. Gasser, P.-E. Jabin and B. Perthame.[7]The equation (1.5) for the perturbation is close the one found in refs. 8 and 9 where R. Glassey and J. Schaeffer study the asymptotic decay of the solutions to a one dimensional linearized Vlasov–Poisson system. Other results of stability around particular solutions to the Vlasov–Poisson system have been obtained by Y. Guo,[10] Y. Guo and G. Rein[11] for steady solutions introduced in ref. 3, and Y. Guo and W. Strauss around homogeneous solutions in a neutral plasma in ref. 12 or around BGK solutions in ref. 13. Another motivation for studying this system comes from systems of particles moving in a Stokes flow where it is natural to study "infinite systems" (see refs. 16, 2 or 21).

2. PROOF OF LEMMA 1.1

We multiply the first equation of (1.4) by $\dot{b}$ and with the second equation

$$\frac{1}{2}\dot{b}^2+\frac{1}{b}=\text{const.} < +\infty \tag{2.1}$$

This shows that b is defined and positive for all time. The first equation of (1.4) implies that a is strictly decreasing. Thus if a is negative initially, it remains so. On the contrary, if a is initially non-negative, then it becomes negative in a finite time since, as long as it is non-negative, b is decreasing and thus $\dot{a}$ also.

3. PROOF OF THE THEOREM 1.1

We recall first the two easy a priori estimates deduced from (1.5) and (1.6)

$$\|f(t,.,.)\|_{L^\infty} \leqslant \|f^0\|_{L^\infty} + \|F\|_{L^\infty}, \qquad f(t,x,v) \geqslant -F(ax+bv) \tag{3.1}$$

To prove the theorem, we consider a sequence f_n of solutions to (1.5) with $\chi_n(x)\,\nabla F(ax+bv)$ instead of $\nabla F(ax+bv)$ in the right-hand side. For χ_n regular enough (in $L^1 \cap L^\infty$), these solutions are known to exist. We then let χ_n converge uniformly toward 1. The problem is to get L^p estimates for E_n, uniformly in n, and then to show that E_n converge strongly toward E.

This section is divided into three parts. First of all, we explain how we can get estimates on ρ from moments of the form $\int (1+|ax+bv|^k)\,|f|^2$ and we show a direct estimate on E in terms of the same moment. Afterwards, we prove that such a moment remains bounded for a finite time independent of n with the assumptions of Theorem 1.1. Finally, we adapt the standard argument to prove the compactness of E_n.

3.1. Estimates on ρ and E

This part is devoted to the proof of two lemmas.

Lemma 3.1. On any time interval $[0,T]$ where $b(t) > 0$, for $k > 3$ and any $f \in L^\infty(\mathbb{R}^6)$, we have for some constant $K(\|f\|_{L^\infty}, (\min b)^{-1})$

$$\left\| \int_{\mathbb{R}^3} f(x,v)\,dv \right\|_{L^{(k+3)/3}} \leqslant K\left(\int (1+|ax+bv|^k)\,|f|^2\,dx\,dv\right)^{3/(k+3)} \tag{3.2}$$

Proof. The argument is classical (see ref. 19 for instance). It makes no difference to work with $1+|ax+bv|^k$ instead of the usual $1+|v|^k$. ∎

This estimate on ρ implies a corresponding estimate on the force field E in some L^p but with $p > 6$. However the system (1.5) has a source term whose decay in the space variable is limited by the decay of E. Hence to propagate a moment of f^2, it is necessary to control the L^2 norm of E, which is done by the lemma

Lemma 3.2. On any tine interval $[0, T]$ where $b(t) > 0$, for any $k > 5$, we have for some constant $K((\min b)^{-1}, \max a)$

$$\int |E_n|^2\,dx \leqslant K \int_0^t \int |E_n|^2\,(s,x)\,dx\,ds$$
$$+K \int_0^t \|E_n(s,.)\|_{L^2} \left(\int (1+|ax+bv|^k)\,|f_n|^2\,dx\,dv \right)^{1/2} ds \qquad (3.3)$$

Proof. This estimate is a consequence of the fact that E_n satisfies the Poisson equation associated with f_n. More precisely, let us compute

$$\int E_n \cdot (ax+bv)\, f_n\,dx\,dv = a \int E_n \cdot x\rho_n\,dx + b \int E_n \cdot j_n\,dx \qquad (3.4)$$

Now since $E_n = \nabla V_n$ with $\Delta V_n = \alpha \rho_n$

$$\int E_n \cdot x\rho_n\,dx = \alpha \int x \cdot \nabla V_n\,\Delta V_n\,dx = -\alpha \int |\nabla V_n|^2\,dx - \alpha \int x \cdot \nabla V_n\,\Delta V_n\,dx$$
$$= -\frac{\alpha}{2} \int |E_n|^2\,dx \qquad (3.5)$$

Whereas the continuity equation implies

$$\int E_n \cdot j_n\,dx = \int j_n \cdot \nabla V_n\,dx = -\int V_n \operatorname{div} j_n\,dx$$
$$= \int V_n\,\partial_t \rho_n\,dx = \alpha \int V_n\,\partial_t\,\Delta V_n\,dx$$
$$= -\frac{\alpha}{2}\frac{d}{dt} \int |\nabla V_n|^2\,dx = -\frac{\alpha}{2}\frac{d}{dt} \int |E_n|^2\,dx \qquad (3.6)$$

Using the system (1.4), we deduce from Eqs. (3.5) and (3.6)

$$\frac{d}{dt}\, b \int |E_n|^2\,dx \leqslant 3\,|a| \int |E_n|^2\,dx + \left| \int E_n \cdot (ax+bv)\, f_n\,dx\,dv \right| \qquad (3.7)$$

It remains to integrate in time and bound this last integral to conclude

$$\left|\int E_n\cdot(ax+bv)\,f_n\,dx\,dv\right| \leqslant \int \frac{|E_n|}{1+|ax+bv|^{k/2-1}}\,(1+|ax+bv|^{k/2})\,|f_n|\,dx\,dv$$
$$\leqslant K\,\|E_n\|_{L^2}\left(\int (1+|ax+bv|^k)\,|f_n|^2\,dx\,dv\right)^{1/2} \tag{3.8}$$

3.2. Propagation of Moments in Short Time

We prove here that there exists a time $t>0$, such that the moment $\int (1+|ax+bv|^k)\,|f_n|^2$ remains finite on $[0,t]$, uniformly in n. To do so, we multiply the equation (1.5) by $(1+|ax+bv|^k)\,f_n$ and we integrate over all $\mathbb{R}^6$. Since a and b are solutions to the system (1.4), we find

$$\frac{d}{dt}\int (1+|ax+bv|^k)\,|f_n|^2\,dx\,dv \leqslant kb\int |ax+bv|^{k-1}\,|E_n|\cdot|f_n|^2\,dx\,dv$$
$$+b\int |E_n|(1+|ax+bv|^k)\,|f_n|\cdot|\nabla F(ax+bv)\,|\,dx\,dv \tag{3.9}$$

Thanks to the hypothesis $(1+|\xi|^k)\,\nabla F\in L^1$, the second term is bounded quite easily

$$b\int |\,E_n|(1+|ax+bv|^k)\,|f_n|\cdot|\nabla F(ax+bv)\,|\,dx\,dv \leqslant K\,\|E_n\|_{L^2}\,\|f_n\|_{L^2} \tag{3.10}$$

To bound the first term in the right hand-side, we first apply the following lemma which is only a more general version of the Lemma 3.1 and is proved the same way.

Lemma 3.3. There exists a constant $K(\|f_n\|_{L^\infty})$ such that

$$\left\|\int (1+|ax+bv|^{k-1})\,|f_n|^2\,dv\right\|_{L^{(k+3)/(k+2)}}$$
$$\leqslant K\left(\int (1+|ax+bv|^k)\,|f_n|^2\,dx\,dv\right)^{(k+2)/(k+3)} \tag{3.11}$$

We deduce that

$$\int |ax+bv|^{k-1}\,|E_n|\cdot|f_n|^2\,dx\,dv$$
$$\leqslant K\,\|E_n\|_{L^{k+3}}\left(\int (1+|ax+bv|^k)\,|f_n|^2\,dx\,dv\right)^{(k+2)/(k+3)} \tag{3.12}$$

Now, we recall that in dimension 3

$$\|E\|_{L^{k+3}} \leqslant K \, \|\rho\|_{L^{3(k+3)/(k+6)}} \tag{3.13}$$

Since $k > 5$, we have the inequality $2 < 3\,\frac{k+3}{k+6} \leqslant \frac{k+3}{3}$ and we can apply the Lemma 3.1 to obtain

$$\|E\|_{L^{k+3}} \leqslant K \left(\int (1+|ax+bv|^k)\, |f_n|^2 \, dx\, dv \right)^{3/(k'+3)} \tag{3.14}$$

with $\frac{k'+3}{3} = 3\,\frac{k+3}{k+6}$. As a consequence, we eventually get the estimate

$$\frac{d}{dt} \int (1+|ax+bv|^k)\, |f_n|^2 \, dx\, dv \leqslant K \, \|E_n\|_{L^2} \, \|f_n\|_{L^2} + K \left(\int (1+|ax+bv|^k)\, |f_n|^2 \, dx\, dv \right)^{\gamma} \tag{3.15}$$

Together with the Lemma 3.2, the Gronwall lemma implies that $\int (1+|ax+bv|^k)\, |f_n|^2$ is bounded on some interval $[0, t]$. Unfortunately the coefficient γ is larger than 1, and therefore we cannot obtain a global bound on the whole interval $[0, T]$.

3.3. Compactness of the Force Field

We now work on the interval $[0, t^*]$ where $\int (1+|ax+bv|^k)\, f_n^2$ is bounded. As a consequence ρ_n belongs to $L^\infty([0, t^*], L^{(k'+3)/3})$ for all $3 < k' \leqslant k$ with a uniform bound. Hence E_n is weakly compact in $L^\infty([0, t^*], L^p)$ for some p. We decompose E_n into the sum of E_n^R and F_n^R with

$$E_n^R = -\left(\frac{\alpha}{4\pi}\, \psi_R(x)\, \frac{x}{|x|^3} \right) \star \rho \tag{3.16}$$

with ψ a C^1 function supported in the annulus $\{1/R \leqslant |x| \leqslant R\}$. Notice here that F_n^R converges toward zero in $L^\infty([0, t], L^p)$ when R vanishes uniformly in n. As for E_n^R, we recall that ρ satisfies the continuity equation

$$\partial_t \rho(t, x) + \operatorname{div} j = 0 \tag{3.17}$$

where $j(t, x) = \int v f(t, x, v)\, dv$ is the current. We can rewrite j as

$$j(t, x) = \int \left(\frac{a}{b}\, x + v \right) f \, dv - \int \frac{a}{b}\, x f \, dv = \tilde{\jmath} - \frac{a}{b}\, x\rho \tag{3.18}$$

The modified current $\tilde{j}$ belongs to some $L^\infty([0,t^*],L^q)$ thanks to the boundedness of the moment, and the term $ab^{-1}x\rho$ to L^r_{loc} with $2<r \leqslant (k+3)/3$. As a consequence $\partial_t\rho$ belongs to $L^\infty([0,t^*],W^{-1,q}+W^{-1,r}_{loc})$. For a fixed R this implies that E_n^R is compact in $L^\infty([0,t^*],L^p)$. We now conclude that it is possible to extract a sub-sequence such that E_n converges in $L^\infty([0,t^*],L^p)$ toward $-\frac{\alpha}{4\pi}\frac{x}{|x|^3}\star\rho$ where ρ is the weak limit of ρ_n. This shows that a weak limit f of f_n satisfies the system (1.5).

4. PROOF OF THE THEOREM 1.2

In this section, we prove the propagation of moments of the form $\int |ax+bv|^k f$ for all $k<k_0$ with $k_0>3$. Since we already know that the L^1 norm of f is bounded, the moment of order k is bounded by the moment of order k' if $k'>k$. Hence, we only consider moment of order $3<k<k_0$. We use arguments similar to those developped in refs. 17 and 7.

Let $\Phi_R\in C^0(B(0,2R))$ be a function with value 1 on $B(0,R)$. We decompose the field E into a short range part E_R and a long range F_R with

$$E_R=\frac{\alpha}{4\pi}\left(\frac{x}{|x|^3}\Phi_R(x)\right)\star\rho \tag{4.1}$$

Now we define the characteristics

$$\begin{aligned}&\dot{X}(s)=-V(s), \qquad \dot{V}(s)=-\alpha cX(s)-F_R(X(s))\\ &(X(0),V(0))=(x,v)\end{aligned} \tag{4.2}$$

The solution f to the system (1.5) can now be written as

$$f(t,x,v)=f^0(X(t),V(t))+\int_0^t \nabla_V(E_Rf+E_RF)(t-s,X(s),V(s))\,ds \tag{4.3}$$

And as a consequence, denoting (Y,W) the inverse of (X,V)

$$\begin{aligned}\rho(t,x)=\rho^0(t,x)&+\operatorname{div}_x\int_0^t\int_{\mathbb{R}^3}(E_R(f+F))(t-s,X(s),V(s))\,\frac{\partial Y}{\partial w}\,dv\,ds\\ &-\int_0^t\int_{\mathbb{R}^3}(E_R(f+F))(t-s,X(s),V(s))\left(\nabla_x\frac{\partial Y}{\partial w}+\nabla_v\frac{\partial W}{\partial w}\right)\end{aligned} \tag{4.4}$$

We now apply the lemma

Lemma 4.1. The following inequalities hold for some constant K

$$\left|\frac{\partial Y}{\partial w}(s)\right|\leqslant Ks, \qquad \left|\nabla_x\frac{\partial Y}{\partial w}(s)\right|\leqslant Ks, \qquad \left|\nabla_v\frac{\partial W}{\partial w}(s)\right|\leqslant Ks \tag{4.5}$$

Proof. Notice that F_R is quite regular, in particular all its derivatives exist and belong to L^∞. Hence all the derivatives of the characteristic field and of the inverse (Y, W) are bounded. All the quantities above are thus lipschitz in time s and since they all vanish at $s=0$ thanks to the initial condition on (X, V), we deduce the lemma. ∎

Thus we would like to control quantities of the form

$$\left\| \int_0^t \int_{\mathbb{R}^3} s(E_R f + E_R F)(t-s, X(s), V(s))\, ds\, dv \right\|_{L^p} \tag{4.6}$$

Let us begin with

$$\begin{aligned} &\left\| \int_0^t \int_{\mathbb{R}^3} s(E_R f)(t-s, X(s), V(s))\, ds\, dv \right\|_{L^p} \\ &\quad \leqslant K \left\| \int_0^t s \left(\int |E_R|^{3/2+\varepsilon} (t-s, X)\, dv \right)^{2/3-\varepsilon} \right. \\ &\qquad \left. \times \left(\int |f|\, (t-s, X, V)\, dv \right)^{1/3+\varepsilon} ds \right\|_{L^p} \\ &\quad \leqslant K \left\| \int_0^t s \left(\int |E_R|^{3/2+\varepsilon} (t-s, x) \left| \det \frac{\partial Y}{\partial w} \right| dx \right)^{2/3-\varepsilon} \right. \\ &\qquad \left. \times \left(\int |f|\, (\ldots)\, dv \right)^{1/3+\varepsilon} ds \right\|_{L^p} \end{aligned} \tag{4.7}$$

We then need the lemma

Lemma 4.2. We have the inequality for a constant K

$$\left| \det \frac{\partial Y}{\partial w} \right| \leqslant \frac{K}{s^3} \tag{4.8}$$

Proof. Choosing R large enough, the characteristic field (X, V) can be made as close as we wish from (X_0, V_0) which satisfies

$$\dot{X}_0 = -V_0, \qquad \dot{V}_0 = -\alpha c X_0, \qquad X_0(0) = x, \qquad V_0(0) = v \tag{4.9}$$

For this last field, we have

$$\frac{\partial \dot{X}_0}{\partial v} = -\frac{\partial V_0}{\partial v}, \qquad \frac{\partial X_0}{\partial v}(0) = 0, \qquad \frac{\partial \dot{V}_0}{\partial v} = -\alpha c \frac{\partial X_0}{\partial v}, \qquad \frac{\partial V_0}{\partial v}(0) = Id \tag{4.10}$$

For $\alpha=1$, we deduce immediately that $\frac{\partial V}{\partial v}$ has all its coefficients on the diagonal increasing (the others are zero). The diagonal coefficients of $\frac{\partial V}{\partial v}$ is given by the value of δ, the solution to the system (1.9). So for $\alpha=-1$, the hypothesis of Theorem 1.2 also ensures that, on the time interval $[0, T]$, $\frac{\partial V}{\partial v}$ is always larger than $K.Id$, with K a positive constant. As a consequence, the lemma is true for X_0 and so for X with R large enough. ▮

Since we already know that E and thus E_R are bounded in $\mathrm{L}^{3/2+\varepsilon}$, we get

$$\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}^3} s(E_R f)(t-s, X(s), V(s))\, ds\, dv \right\|_{\mathrm{L}^p} \\
&\quad \leqslant K \left\| \int_0^t s^{-1+\varepsilon} \left(\int |f|\,(t-s, X(s), V(s))\, dv \right)^{1/(3+\varepsilon)} ds \right\|_{\mathrm{L}^p} \\
&\quad \leqslant K \sup_t \left(\int |aX+bV|^{p-3+\varepsilon}\, |f|\,(\ldots)\, dv\, dx \right)^{1/p} \\
&\quad \leqslant K \sup_t \left(\int |ax+bv|^{p-3+\varepsilon}\, |f|\,(t-s, x, v)\, dv\, dx \right)^{1/p}
\end{aligned} \tag{4.11}$$

since, if $|aX+bV|$ is bounded by L, then v is in a ball of radius of order L. This is due to the fact that $\frac{\partial V}{\partial v}$ can be very close to $K.Id$ with K positive and bounded for R large enough. With the same argument we obtain that

$$\begin{aligned}
&\left\| \int_{t_0}^t \int_{\mathbb{R}^3} s(E_R f)(t-s, X(s), V(s))\, ds\, dv \right\|_{\mathrm{L}^p} \\
&\quad \leqslant K \log(t_0) \sup_t \left(\int |ax+bv|^{p-3}\, |f|\,(t-s, x, v)\, dv\, dx \right)^{1/p}
\end{aligned} \tag{4.12}$$

We denote q such that $1/p+1=2/3+1/q$ and p^* the conjugate exponent of p. Let us now turn to

$$\begin{aligned}
&\left\| \int_0^t \int_{\mathbb{R}^3} sE_R(t-s, X)\, F(a(t-s)\, X(s)+b(t-s)\, V(s))\, ds\, dv \right\|_{\mathrm{L}^p} \\
&\quad \leqslant \left(\int_0^t s \left(\left(\int |E_R|^{1-p^*/2p}\,(t-s, X)\, F^{p^*-qp^*/p}(a(.)\, X(s)+b(.)\, V(s))\, dv \right)^{p/p} \right.\right. \\
&\qquad \left.\left. \times \int_{\mathbb{R}^3} |E_R|^{3/2}\,(t-s, X)\, F^q(a(.)\, X(s)+b(.)\, V(s))\, ds\, dv\, dx \right)^{1/p}
\end{aligned} \tag{4.13}$$

Of course, we have

$$\int |E_R|^{1-p^*/2p}\,(t-s,\,X)\,F^{p^*-qp/p^*}(a(.)\,X(s)+b(.)\,V(s))\,dv$$

$$\leqslant \left(\int |E_R|^{3/2}\,(\ldots)\,dv\right)^{2p^*/3p-p^*/p^2}\left(\int F^q(\ldots)\,dv\right)^{p^*/pq-p^*/p^2}$$

$$\leqslant \left(\int |E_R|^{3/2}\,(t-s,\,x)\,dx\left|\det\frac{\partial Y}{\partial w}\right|\right)^{2p^*/3p-p^*/p^2}$$

$$\times\left(\int F^q(a(.)\,X(V^{-1})+b(.)\,v)\left|\det\frac{\partial W}{\partial w}\right|dv\right)^{p^*/pq-p^*/p^2}$$

$$\leqslant Ks^{(3/p-2)\,p^*/p} \tag{4.14}$$

since, as we have already seen, $\det\frac{\partial Y}{\partial w}$ is less than Ks^{-3} and $det\,\frac{\partial W}{\partial w}$ is bounded on the time interval we consider. We also know that

$$\sup_t\int_{\mathbb{R}^6}|E_R|^{3/2}\,(t-s,\,X)\,F^q(a(.)\,X(s)+b(.)\,V(s))\,dv\,dx\leqslant K \tag{4.15}$$

Finally, we obtain

$$\left\|\int_0^t\int_{\mathbb{R}^3}sE_R(t-s,\,X)\,F(a(t-s)\,X(s)+b(t-s)\,V(s))\,ds\,dv\right\|_{\mathrm{L}^p}$$

$$\leqslant K\left(\int_0^t s^{3/p-1}\,ds\right)^{1/p}\leqslant K \tag{4.16}$$

since $3/p>0$. The last term with ρ^0 is rather easy to control

$$\|\rho^0(t,\,.)\|_{\mathrm{L}^q}=\left(\int\left(\int f^0(X,V)\,dv\right)^q dx\right)^{1/p}$$

$$\leqslant K\left(\int |aX+bV|^{3q-3}\,|f^0|\,(X,V)\,dx\,dv\right)^{1/p} \tag{4.17}$$

because, as before, $\frac{\partial V}{\partial v}$ is as close as we want to $K.Id$ with K positive and bounded, as explained in the Lemma 4.2. Now of course to have E_R in L^p, we need ρ^0 in L^q with $1/p=1/q-1/3$, which implies that $3q-3\leqslant p-3$ for $q\geqslant 2$ or $p\geqslant 6$. To conclude, if we put all these informations together, we obtain that for $k>3$ and for any t_0, if $t\leqslant t_0$

$$\|E_R(t,\,.)\|_{\mathrm{L}^{k+3}}\leqslant K+K\sup_{s\leqslant t}\left(\int |ax+bv|^{k+\varepsilon}\,|f|\,(s,\,x,\,v)\,dx\,dv\right)^{1/(k+3)} \tag{4.18}$$

and if $t > t_0$

$$\|E_R(t,.)\|_{L^{k+3}} \leqslant \|E_R(t_0,.)\|_{L^{k+3}} + K\log(t_0) \times \sup_{t>s>t_0}\left(\int |ax+bv|^k\,|f|\,(t,x,v,)\,dx\,dv\right)^{1/(k+3)} \tag{4.19}$$

It is now necessary to express the moments in term of the L^p norm of E, which is done by the lemma

Lemma 4.3. Suppose that $|\xi|^k\,|\nabla F| \in L^1$ and $E \in L^1([0,T]\times\mathbb{R}^3)$. Then on the inverval $[0,T]$, we have

$$\int |ax+bv|^k\,|f|\,(t,x,v)\,dx\,dv \leqslant K + K\int |a(0)\,x+b(0)\,v|^k\,|f^0|\,dx\,dv + K\left(\int_0^t \|E\|_{L^{k+3}}\,ds\right)^{k+3} \tag{4.20}$$

Proof. This lemma is the precise equivalent of the usual moment lemma for Vlasov–Poisson. It is also proved by multiplicating the equation by $|ax+bv|^k$. ∎

We propagate the moments in two steps: first, we justify a short time propagation and find an estimate for $\|E_R\|$ at t_0 thanks to formula (4.18) and then we use formula (4.19) to prove it globally in time.

Lemma 4.4. There exists t_0 and ε such that

$$\int_{\mathbb{R}^6} |ax+bv|^{k+\varepsilon}\,|f|\,(t,x,v)\,dx\,dv \in L^\infty([0,t_0]) \tag{4.21}$$

Proof. Let us apply the Lemma 4.3 to the moment of order $k+3+\varepsilon$. For ε small enough, this is less than k_0 and so this moment is initially finite, then

$$\int |ax+bv|^{k+\varepsilon}\,|f|\,(t,x,v)\,dx\,dv \leqslant K + K\left(\int_0^t \|E\|_{L^{k+3+\varepsilon}}\,ds\right)^{k+3} \tag{4.22}$$

Using now Young inequality and a well known estimate, we write

$$\|E(s,.)\|_{L^{k+3+\varepsilon}} \leqslant K\,\|\rho\|_{L^{(l+3)/3}} \leqslant K\left(\int |ax+bv|^l\,|f|\,dx\,dv\right)^{3/(l+3)} \tag{4.23}$$

with $1/(k+3+\varepsilon)=3/(l+3)-1/3$. Since $k>3$, for ε small enough, it is possible to have $l\leqslant k$. For such an ε, we obtain for t bounded

$$\int |ax+bv|^{k+\varepsilon}\,|f|\,(t,x,v)\,dx\,dv\leqslant K+K\left(\int_0^t\int |ax+bv|^k\,|f|\,dx\,dv\right)^{\alpha} \tag{4.24}$$

with α a real number larger than 1. From this, we deduce the lemma. ▮

The formula (4.18) implies that there exists t_0 such that $\|E_R(t_0,.)\|_{L^{k+3}}$ and $\int |ax+bv|^k\,|f|\,(t_0,.,.)$ are finite. To propagate after the time t_0, we combine the formula (4.19) and the Lemma 4.3 to get for t bounded

$$\int |ax+bv|^k\,|f|\,(t,x,v)\,dx\,dv\leqslant K+K\int_{t_0}^t\int |ax+bv|^k\,|f|\,(s,x,v)\,dx\,dv\,ds \tag{4.25}$$

This shows that the moment $\int |ax+bv|^k\,|f|$ remains finite for all times in $[0,T]$, thus concluding the proof of the Theorem 1.2.

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